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Phase transitions in the gauge invariant random Potts model

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Abstract. The duality transformation is given for the gauge invariant random Potts model on a square lattice. Gauge invariance and duality properties of the partition function are used to demonstrate the existence of a frustration-dissociation phase transition in the ground state at a critical concentration of ferromagnetic bonds. The phase diagram is obtained with the aid of an annealed approximation.

1. Introduction

The gauge symmetries of random Ising and xy models in two and three dimensions have been discussed by various authors [1, 2, 3]. The frustration function, introduced by Toulouse [4], provides an elegant, gauge invariant description of the relevant disorder in such systems [1]. The quenched partition function is found to depend only on the distribution of frustrations, and may be written as an annealed partition function in the presence of a plaquette coupling term, the limit of infinitely strong plaquette couplings fixing a given configuration of frustrations [1, 2]. In the case of the two-dimensional random Ising model, the duality transformation performed on the partition function (before taking the limit of infinitely strong plaquette couplings) provides a way of expressing the partition function in the presence of n frustrations in terms of an n-point correlation function of the unfrustrated, zero-field Ising model.

Schuster [5] has noted that in this case, the probability weights of the different frustration configurations in the quenched average may be similarly expressed in terms of the correlation functions of the dual unfrustrated model, where the temperature is now a function of the concentration of ferromagnetic bonds, x. He has shown that a phase transition with a local order parameter takes place in the frustration system at some concentration x_c. One finds $x_c < x_0$, where x_0 is the concentration at which the ferromagnetic transition is destroyed. Thus, two qualitatively different types of disordered state at $T = 0$ are predicted for the random ($\pm J$) Ising model in two dimensions.

In this paper the above treatment is generalised to the q-state vector Potts model. An appropriate frustration function is defined, which for $q = 2$ is equivalent to that of Toulouse [4]. The duality transformation on the partition function in the presence of n frustrations gives the n-point correlation functions of the unfrustrated Potts model on the dual lattice. In the appendix the duality transformation is given in detail for an annealed representation of the frustrated partition function, with plaquette coupling terms.

The critical concentration of 'ferromagnetic' bonds, x_c, where the frustration-dissociation transition occurs, is found to be

$$x_c = q^{-1/2},$$

which reduces to Schuster's result [5] for $q = 2$. It should be noted that this transition becomes *first order* for $q \geqslant 4$. To obtain an annealed approximation to x_0, the concentration at which the order-disorder transition in the random q-state vector Potts model is depressed to $T = 0$, we have performed an exact de-decoration transformation on a decorated lattice *à la* Syozi [6]. This calculation turns out to be exactly equivalent to that of Sarbach and Wu [7] and yields

$$x_0 = \tfrac{1}{2}(1 + q^{-1/2}).$$

Comparison with the Monte Carlo results of Vannimenus and Toulouse [8] for the Ising model shows that the annealed approximation to x_0 is yet lower than the numerical result for the quenched case, so that the possibility of a frustration-unbinding transformation in the ordered phase of the random Potts system seems to be ruled out. On the other hand, re-expressing the probability weights in the quenched average for the free energy in terms of n-point correlation functions, we see that the free energy is singular at $x = x_c$. This fact has already been pointed out by Schuster [5]. We would like to underline, however, that this singularity persists at *all* temperatures, and is the analogue as a function of x, of the Griffiths singularities [9], encountered in random systems at the critical temperature of the pure system, and which persist at all concentrations.

2. The model

Consider the general Z_q symmetric Hamiltonian [10, 11]

$$\mathcal{H} = \frac{1}{q} \sum_{\langle ij \rangle} \sum_{\mu=1}^{q} J_{ij}^{\mu} (S_i^* S_j)^{\mu} \tag{1}$$

where $S_i = \exp[(2\pi\mathrm{i}/q)m_i]$, $m_i = 1, 2, \ldots, q$. If we choose $J_{ij}^{\mu} = J \exp(2\pi \mathrm{i} r_{ij}\mu/q)$, $r_{ij} = 0, \ldots, q-1$, we obtain the random gauge invariant Potts model [12, 13]. The pure Potts model is the case where all $r_{ij} = 0$, and the Z_q clock model is obtained when $J^{\mu} = 0$ for $\mu \neq 1$. The partition function for the random Potts model is then,

$$Z_K = \sum_{\{S_i\}} \exp\left(\frac{K}{q} \sum_{\langle ij \rangle} \sum_{\mu} (S_i^* A_{ij} S_j)^{\mu}\right) \tag{2}$$

where $K = \beta J$, $A_{ij} = \exp(2\pi\mathrm{i} r_{ij}/q)$. Z_K is invariant under the set of local transformations

$$A'_{ij} = M_i A_{ij} M_j^*$$
$$M_i = \exp(2\pi\mathrm{i}\lambda_i/q) \qquad \lambda_i = 0, \ldots, q-1 \tag{3}$$

and the Hamiltonian is invariant under the transformations (3) with

$$S'_i = M_i S_i. \tag{4}$$

We define the gauge invariant plaquette functions [14, 15]

$$
\begin{aligned}
A_p &= \prod_{\langle ij\rangle < p} A_{ij} = \exp(2\pi \mathrm{i} r_p/q) \\
r_p &= \operatorname{mod}_q \sum_{\langle ij\rangle < p} r_{ij} \qquad r_p = 0, \ldots, q-1.
\end{aligned}
\tag{5}
$$

A plaquette is said to be frustrated for $r_p \neq 0$, [16].

2.1. Duality transformations

The duality transformations for Z_q symmetric models on a square lattice were given by Wegner [17]†. Using his results we immediately find the following duality relation for the partition function in the presence of a frustration configuration specified by $\{r_p\}$, normalised by the partition function in the absence of frustrations:

$$
\frac{Z_K\{r_p\}}{Z_K\{r\equiv 0\}} = \left\langle \prod_{\tilde{\imath}} S_{\tilde{\imath}}^{r_p} \right\rangle_{\tilde{K}}. \tag{6}
$$

The correlation function is that of a uniform Potts model on the dual lattice, with coupling constants given by

$$
(\mathrm{e}^{\tilde{K}} - 1)(\mathrm{e}^{K} - 1) = q \tag{7}
$$

and r_p are the frustrations at the plaquettes p dual to the sites $\tilde{\imath}$. We can express the correlation function in a more familiar form, if we take, e.g., the case of two frustrations at the plaquettes 1 and 2, with $r_1 = -r_2 = r$. Then

$$
\left\langle \sum_{r=1}^{q-1} (S_1^* S_2)^r \right\rangle_{\tilde{K}} = \langle q\delta_{S_1,S_2} - 1\rangle_{\tilde{K}} \tag{8}
$$

the two-point correlation function of the Potts model. We give an alternate derivation of the duality relation in the appendix using the graphical expansion of the partition function. From there we learn that there is a further symmetry of the partition function, namely, that $Z_K\{r_p = -r_q, r_s = 0, s \neq p, q\}$ does not depend explicitly on the values of r_p. Thus

$$
\frac{1}{q-1}\langle q\delta_{S_1,S_2} - 1\rangle_{\tilde{K}} = \frac{Z_K\{r_1 = -r_2;\, r_p = 0;\, p \neq 1, 2\}}{Z_K\{r \equiv 0\}}. \tag{9}
$$

2.2. Quenched averages

The gauge invariance of the plaquette functions can be used to rewrite the quenched partition function in a way that makes the frustration dependence explicit [1, 18]. We may write, up to an infinite constant,

$$
Z_K\{r_p\} = \lim_{K_p\to\infty} q^{-N} \sum_{\{A_{ij}\}} \sum_{\{S_i\}} \exp\left(\frac{K}{q} \sum_{\langle ij\rangle} \sum_{\mu} (S_i^* A_{ij} S_j)^{\mu} + \frac{K_p}{q} \sum_{p} \sum_{\mu} (\phi_p^* A_p)^{\mu} \right) \tag{10}
$$

where $\phi_p = \exp[(2\pi\mathrm{i}/q) r_p]$. We can, moreover, use the gauge invariance of the spin

† See also [10, 11] and [25].

term to fix the spins $S_i \equiv 1$ and write,

$$Z_K\{r_p\} = \lim_{K_p \to \infty} \sum_{\{A_{ij}\}} \exp\left(\frac{1}{q} K \sum_{\langle ij \rangle} \sum_{\mu} (A_{ij})^\mu + \frac{1}{q} K_p \sum_p \sum_\mu (\phi_p^* A_p)^\mu\right). \tag{11}$$

The quenched averages over thermodynamic quantities can now be performed via [5]

$$\bar{Q} = \sum_{\{\phi_p\}} P\{\phi_p\} Q\{\phi_p\}.$$

In particular, Q may be the free energy, $-\beta F\{\phi_p\} = \ln Z\{\phi_p\}$. The probability weights for the frustration configurations can be obtained from the respective weights for the bond configurations [5]

$$P\{\phi_p\} = \sum_{\{A_{ij}\}} P\{A_{ij}\} \prod_p \delta\{\phi_p, A_p\}. \tag{12}$$

Again up to an infinite constant

$$P\{\phi_p\} = \lim_{K_p \to \infty} \sum_{\{A_{ij}\}} P\{A_{ij}\} \exp\left(\frac{1}{q} K_p \sum_p \sum_\mu (\phi_p^* A_p)^\mu\right). \tag{13}$$

Assuming that the bonds are distributed independently, according to

$$p(A_{ij}) = x\delta(A_{ij} - 1) + \frac{1-x}{q-1} \sum_{r=1}^{q-1} \delta(A_{ij} - \exp(2\pi \mathrm{i}/q) r) \tag{14}$$

we may write

$$P\{A_{ij}\} = \prod_{\langle ij \rangle} p(A_{ij}) = \alpha^E \exp\left(K_\mathrm{F} \sum_{\langle ij \rangle} \sum_\mu (A_{ij})^\mu / q\right) \tag{15}$$

where E is the number of edges on the lattice, and K_F and α are found from (14) to be

$$\begin{aligned} \alpha &= (1-x)/(q-1) \\ K_\mathrm{F} &= \ln[x(q-1)/(1-x)]. \end{aligned} \tag{16}$$

Substituting (14) into (13) and comparing with (11) we obtain,

$$\frac{P\{\phi_p\}}{P\{\phi \equiv 1\}} = \frac{Z_{K_\mathrm{F}}\{\phi_p\}}{Z_{K_\mathrm{F}}\{\phi \equiv 1\}}. \tag{17}$$

(Notice that here we use the sets $\{\phi_p\}$ and $\{r_p\}$ interchangeably.) Furthermore, using the duality relation, (6), we have,

$$\frac{P\{\phi_p\}}{P\{\phi \equiv 1\}} = \left\langle \prod_{\tilde{i}} S_{\tilde{i}}^{r_p} \right\rangle_{\tilde{K}(K_\mathrm{F})}. \tag{18}$$

In particular (see appendix, (A10))

$$\frac{P\{\phi_p \neq 1;\, \phi_q = 1;\, q \neq p\}}{P\{\phi \equiv 1\}} = \frac{1}{q-1} \langle q\delta_{S_{\tilde{i}},1} - 1 \rangle_{\tilde{K}(K_\mathrm{F})} \tag{19}$$

where the right-hand side is just the order parameter for a uniform Potts model at the effective coupling constant $\tilde{K}(K_\mathrm{F})$. The frustration system is then seen to have a frustration-dissociation transition [5] at the critical temperature of the Potts model given by the self-duality relation $\tilde{K}(K) = K$. The probability of encountering isolated frustrations vanishes above some critical concentration x_c, or, for $K_F > K_F(x_c)$. Using (16) and (7) we find

$$\chi_c = q^{-1/2}. \tag{20}$$

For $q=2$, this result reduces to that obtained by Schuster [5] for the Ising model. Notice that the transition is *first order* for $q \geqslant 4$.

The free energy of the quenched random model,

$$-\overline{\beta F(K, x)} = \langle \sum_{\{\phi_p\}} P\{\phi_p\} \ln Z\{\phi_p\} \tag{21}$$

can now be written in terms of the correlation functions of the Potts model:

$$-\overline{\beta F(K, x)} = Z_{K_F(x)}\{\phi \equiv 1\} \sum_{\{\phi_p\}} \left\langle \prod_i S_i^{r_p} \right\rangle_{\tilde{K}[K_F(x)]} \ln Z_K\{\phi_p\}. \tag{22}$$

The free energy as a function of the concentration x will have singularities at x_c coming from the singularities of the correlation functions in (22), at the Potts critical point $\tilde{K}(x_c)$, independently of the temperature, i.e. K.

3. The phase diagram

It is interesting to compare x_c with the value x_0 of the concentration of ferromagnetic bonds at which the frustrated plaquettes percolate, suppressing the transition temperature for the order-disorder transition in the Potts model to zero. For $q=2$, Vannimenus and Toulouse [8] have obtained the numerical value of $x_0 = 0.91$ for the square lattice. For general q we may get an approximate value by calculating the corresponding quantity for the annealed random vector Potts model. To this end we have generalised the de-decoration transformation of Syozi [6] for the Ising model. We first replace each bond on the square lattice by a set of decorated bonds in parallel, as shown in figure 1. Each of the intermediate spin sites may be occupied exclusively of the others. Introducing a chemical potential ξ for the occupation of the first intermediate site giving rise to an effective ferromagnetic interaction,

$$x = \partial \ln \Xi / \partial \xi. \tag{23}$$

Performing the summations over the intermediate spins, the grand canonical partition function is found to be

$$\Xi = \exp(EC) Z(K)$$

where E is the number of edges on the underlying lattice, C is the free energy per bond contributed by the intermediate spins, $Z(K)$ the partition function of the uniform Potts model on the underlying lattice, and C and K are given in terms of ξ and the coupling constant on the decorated bonds, L. The critical value of x is then calculated, as a function of L, the coupling constant for the annealed vector Potts model, from a knowledge of the critical values of K and

$$\varepsilon = E^{-1} \partial \ln Z(K) / \partial K$$

the nearest-neighbour correlation function of the underlying Potts model [19]. The calculation reduces exactly to that of Sarbach and Wu[7], who treated the annealed

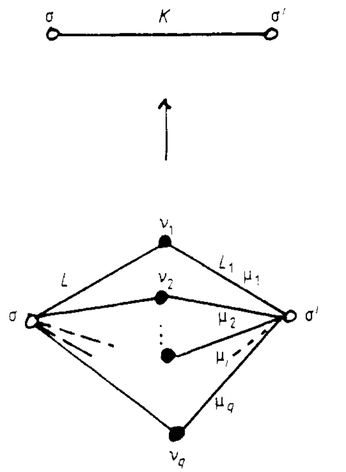

Figure 1. Bond decoration scheme for the annealed random Potts model. The partition function for the decorated bond is

$$\exp(C + K\delta_{\sigma\sigma'}) = \sum_{\nu_1,\nu_2\ \nu_q=1}^{q} \exp\left(h \sum_{i=1}^{q} (\delta_{\sigma,\nu_i} + \delta_{\nu_i+\mu_i,\sigma'}) + \sum_{i=1}^{q} \delta_{\nu_{1,i}} \right)$$

$$\mu_k = k-1,\ \nu_i + \mu_i = \mathrm{mod}_q(\nu_i + \mu_i).$$

random Potts model with a bond distribution given by

$$p(K) = x\delta(K - K_0) + (1-x)\delta(K + K_0).$$

The concentration x_0 at which the critical coupling $L_c \to \infty$ ($T_c \to 0$) is given by

$$x_0 = \varepsilon_c = \tfrac{1}{2}(1 + q^{-1/2}) \tag{24}$$

for $q \leqslant 4$. (Above $q = 4$, the phase transition on the underlying lattice becomes first order, giving $\varepsilon(T_c^+) \neq \varepsilon(T_c^-)$ thus yielding two limiting concentrations, which seems to be unphysical [7].)

Comparing the numerical value [6, 7] for $q = 2$, $x_0 = 0.854$, with the value from the Monte Carlo calculation [8] for the quenched system, we see that the annealed value is less than the numerical approximation to the quenched case. Comparing (22) and (24) we see that for $q \neq 1$ and finite,

$$x_c < x_0^{\text{annealed}}$$

and probably $x_0^{\text{annealed}} < x_0^{\text{quenched}}$ in this range of q. The phase diagram is shown qualitatively in figure 2. There is no possibility of the frustration–dissociation transition preceding the destruction of the ordered phase, thus giving rise to two different types of order.

4. Remarks

Our treatment in § 2 carries over immediately also to a random Z_q 'clock model', with

$$\beta\mathscr{H} = K \sum_{\langle ij\rangle} (S_i^* A_{ij} S_j + \mathrm{cc}).$$

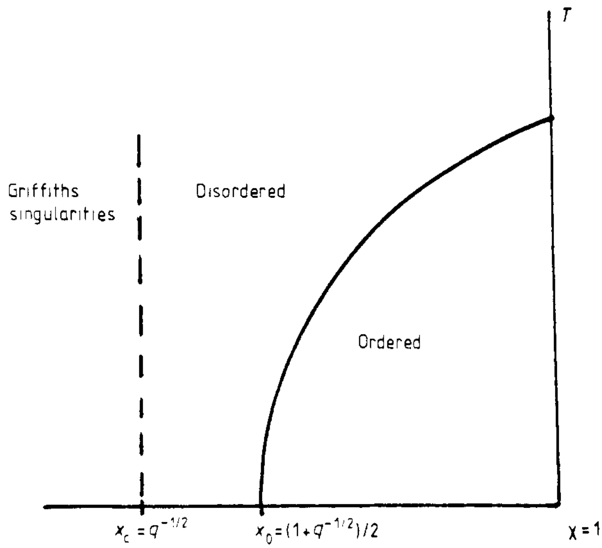

Figure 2. Schematic phase diagram for the two-dimensional quenched random vector Potts model on a square lattice. The value of x_c is exact, whereas x_0 is obtained in an annealed approximation.

However, for $q > 4$ this model is not self-dual, so that we have to look for other ways of locating the transition point than the self-duality relation [10, 11]. For $q > 4$ the situation is more interesting with the appearance of a 'massless' intermediate phase between the ordered and disordered phases [10, 11]—here, a region where the probability of finding two frustrations at a distance l from each other decays algebraically with l. The transition between this and the 'ordered' state (at a smaller concentration x of 'ferromagnetic' bonds) is due to a condensation of 'strings' of bonds with $A_{ij} \neq 1$ [20]. Work is in progress to obtain analogous phase diagrams for *frustrated* spin and gauge Z_q models [21, 22, 23] in higher dimensionalities.

Acknowledgments

I would like to thank Professor F J Wegner and H Horner for extremely useful discussions and Dr W Kinzel for bringing the work of Sarbach and Wu [7] to my attention.

Appendix

The Whitney polynomial [19] expansion of (10) before taking the limit of infinite plaquette coupling is found to be

$$Z_{K_1 K_P}\{r_p\} = q^E \sum_{G, G_p} \left(\frac{u}{q}\right)^{b(G)} \left(\frac{v}{q}\right)^{p(G)} q^{c_1(G, G_p)} \prod_{c' \in G} \delta\left(\sum_{p \in c'} r_p\right) \tag{A1}$$

where $u = e^K - 1$, $v = e^{K_P} - 1$, E is the number of edges on the lattice.

G = graphs on the lattice consisting of bonds and vertices which are not incident to any bond
G_p = graphs on the lattice consisting of plaquettes
$b(g)$ = number of bonds in G
$p(G_p)$ = number of plaquettes in G_p
$c_1(G, G_p)$ = number of circuits in G completely filled (spanned) by plaquettes in G_p.

The sum $\Sigma_{p\in c'}$ runs over all circuits in G spanned by plaquettes in G_p.

Consider the unfrustrated random Potts model with a random field on the dual lattice,

$$-\beta\tilde{\mathscr{H}} = \frac{\tilde{K}}{q}\sum_{\langle\tilde{i}\tilde{j}\rangle}\sum_{\mu}(S_{\tilde{i}}\Psi_{\tilde{i}\tilde{j}}S_{\tilde{j}})^{\mu} - \frac{h}{q}\sum_{\tilde{i}}\sum_{\mu}(S_{\tilde{i}}^{*}\Gamma_{\tilde{i}})^{\mu} \tag{A2}$$

where $S_{\tilde{i}} = \exp[(2\pi i/q)m_{\tilde{i}}]$ and $\Psi_{\tilde{i}\tilde{j}} = A_p^{*}A_q$, with $\tilde{i}, \tilde{j}$ dual sites to the plaquettes p, q. The A_p are given by (5), and we chose the $\Gamma_{\tilde{i}} = A_p^{*}$. The Hamiltonian is also invariant under the transformations in (3) and (4). The partition function has the graphical expansion

$$Z_{\tilde{K},h}\{r_p\} = \sum_{\tilde{G},G_k}\tilde{u}^{b(\tilde{G})}w^{p(G_k)}q^{m(\tilde{G},G_k)} \tag{A3}$$

where $\tilde{u} = e^{\tilde{K}} - 1$, $w = e^{h} - 1$. $\tilde{G}$ and G_k are dual graphs to G and G_p, obtained respectively by (i) placing bonds rotated by 90° on links not occupied by bonds in G, and (ii) placing points at the centres of plaquettes not occupied by plaquettes in G_p.

m = number of connected parts in $\tilde{G}$ that do not have any vertices in common with G_k
$m = c_1(G, G_p)$
$b(\tilde{G}) = E - b(G)$
$p(G_k) = N - p(G_p)$.

Notice that the right-hand side of (A3) does not depend upon $\{r_p\}$. In fact, the model in (A2) is a *generalised Mattis* spin glass. With the redefinition

$$A_pS_{\tilde{i}} \equiv \tau_{\tilde{i}} \tag{A4}$$

the Hamiltonian becomes

$$-\beta\tilde{\mathscr{H}} = \frac{\tilde{K}}{q}\sum_{\langle\tilde{i}\tilde{j}\rangle}\sum_{\mu}(\tau_{\tilde{i}}^{*}\tau_{\tilde{j}})^{\mu} - \frac{h}{q}\sum_{\tilde{i}}\sum_{\mu}(\tau_{\tilde{i}}^{*}\Gamma_0)^{\mu} \tag{A5}$$

where $\Gamma_0 = 1$. This model therefore has a phase transition at $\tilde{K}_c$ given by $e^{\tilde{K}_c} - 1 = q^{1/2}$ for $h = 0$.

(1) Consider the expectation value

$$\frac{1}{q-1}\left\langle\sum_{\mu=1}^{q-1}(S_{\tilde{i}}^{*}\Gamma_{\tilde{i}})^{\mu}\right\rangle_{\tilde{K},h} = \frac{1}{Z_{\tilde{K},h}}\sum_{\tilde{G},G_k}{}'\,\tilde{u}^{b(\tilde{G})}w^{p(G_k)}q^{m(\tilde{G},G_k)}. \tag{A6}$$

The prime indicates a restriction to graphs such that the site $\tilde{i}$ is part of a connected subgraph with at least one vertex in common with the graph G_k. With $r_p \neq 0$, and using (A1), (A3) and (A6), we find

$$\frac{1}{q-1}\left\langle\sum_{\mu=1}^{q-1}(S_{\tilde{i}}^{*}\Gamma_{\tilde{i}})^{\mu}\right\rangle_{\tilde{K},h}=\frac{Z_{K,K_p}\{r_p\neq 0;\text{ all } r_q=0, q\neq p\}}{Z_{K,K_p}\{r\equiv 0\}} \tag{A7}$$

where p is dual to $\tilde{i}$, given that $\tilde{u}u=q$, $vw=q$, or

$$\tilde{K}=-\ln\left(\frac{e^{K}-1}{e^{K}+q-1}\right) \tag{A8}$$

$$h=-\ln\left(\frac{e^{K_p}-1}{e^{K_p}+q-1}\right). \tag{A9}$$

The duality relation given in (A8) is already well known for the uniform lattice [19] and has also been derived for the quenched random Potts model by Jauslin and Swendson [24], but their derivation holds only in the case of positive random couplings, whereas here we have included 'non-ferromagnetic' interactions.

Notice that from (A9) we have $\lim_{K_p\to\infty} h=0$. Thus,

$$\frac{Z_K\{r_p\neq 0;\ r_q=0;\ q\neq p\}}{Z_K\{r\equiv 0\}}=\frac{1}{q-1}\langle q\delta_{\tau_i,1}-1\rangle_{\tilde{K}} \tag{A10}$$

where the RHS is the order parameter ('magnetisation') for the uniform Potts model in zero field. (We have used equation (A4).) Note also that the configuration with a single frustration as above calls for the insertion of an infinite number of 'dislocations' [17] going out to the edge of the lattice, and therefore cannot be arrived at using the results of [17].

(2) Consider the case r_p, $r_q\neq 0$, all other $r_s=0$.

$$\begin{aligned} Z_{K,K_p}&\{r_p, r_q\neq 0;\ r_s=0,\ s\neq p, q\}\\ &=q^{E}\left(\sum_{G,G_p}^{A}\Xi_{G,G_p}\delta(r_p+r_q)+\sum_{G,G_p}^{B}\Xi_{G,G_p}\delta(r_p)\delta(r_q)\right.\\ &\quad\left.+\sum_{G,G_p}^{C}\Xi_{G,G_p}[\delta(r_p)+\delta(r_q)]+\sum_{G,G_p}^{D}\Xi_{G,G_p}\right)\\ &\Xi_{G,G_p}=\left(\frac{u}{q}\right)^{b(G)}\left(\frac{v}{q}\right)^{p(G)}q^{c_1(G,G_p)}. \end{aligned} \tag{A11}$$

The restricted graphical sums are defined so that

(A) the sum is over all graphs such that the plaquettes p, q fall within the same connected graph in G_p, spanning a closed circuit in G;

(B) p and q fall within different connected graphs in G_p, both spanning closed circuits in G;

(C) either p *or* q fall within such graphs;

(D) neither p nor q fall within such graphs.

$\delta(r_p)=\delta(r_q)=0$ by hypothesis. Thus we have,

$$Z_{K,K_p}\{r_p, r_q\}=q^{E}\left(\sum_{G,G_p}^{A}\delta(r_p+r_q)\Xi_{G,G_p}+\sum_{G,G_p}^{D}\Xi_{G,G_p}\right). \tag{A12}$$

Now consider the correlation function in the dual system,

$$\frac{1}{q-1}\left\langle \sum_{\mu=1}^{q-1} (S_{\tilde{i}}^{*}\Gamma_{\tilde{i}})^{\mu} \sum_{\nu=1}^{q-1} (S_{\tilde{i}}^{*}\Gamma_{\tilde{j}})^{\nu} \right\rangle_{\tilde{K},h}$$

$$= \frac{1}{q-1}\frac{1}{Z_{\tilde{K},h}}\left(\sum_{\tilde{G},G_K}^{\tilde{A}} (q-1)\tilde{\Xi}_{\tilde{G},G_k} + \sum_{\tilde{G},G_k}^{\tilde{B}} \tilde{\Xi}_{\tilde{G},G_K}\cdot 0 + \sum_{\tilde{G},G_K}^{\tilde{C}} \tilde{\Xi}_{\tilde{G},G_K}\cdot 0 + \sum_{\tilde{G},G_K}^{\tilde{D}} (q-1)^2 \tilde{\Xi}_{\tilde{G},G_K}\right) \tag{A13}$$

where

$$\tilde{\Xi}_{\tilde{G},G_K} = \tilde{u}^{b(\tilde{G})} w^{p(G_k)} q^{m(\tilde{G},G_K)}$$

and the restricted sums are over graphs $\tilde{G}$, G_k such that:

($\tilde{\mathrm{A}}$) $\tilde{i}, \tilde{j}$ fall on the same connected subgraph in $\tilde{G}$. This subgraph has no vertices in common with G_k;

($\tilde{\mathrm{B}}$) $\tilde{i}, \tilde{j}$ fall on different connected subgraphs of $\tilde{G}$, neither of which have any points in common with G_k;

($\tilde{C}$) $\tilde{i}, \tilde{j}$ fall on different connected subgraphs of $\tilde{G}$, one of which has at least one vertex in common with G_k;

($\tilde{\mathrm{D}}$) $\tilde{i}, \tilde{j}$ fall on connected subgraph(s) of $\tilde{G}$ that has (have) at least one vertex in common with G_k.

Equation (A13) becomes,

$$\frac{1}{q-1}\left\langle \sum_{\mu=1}^{q-1} (S_{\tilde{i}}^{*}\Gamma_{\tilde{i}})^{\mu} \sum_{\nu=1}^{q-1} (S_{\tilde{j}}^{*}\Gamma_{\tilde{j}})^{\nu} \right\rangle = \frac{1}{Z_{\tilde{K},h}}\left(\sum_{\tilde{G},G_K}^{\tilde{A}} \cdot + (q-1) \sum_{\tilde{G},G_K}^{\tilde{D}} \cdot \Xi_{\tilde{G},G_K}\right). \tag{A14}$$

Now observe that the sets of graphs A, B, C ... are dual to $\tilde{\mathrm{A}}$, $\tilde{\mathrm{B}}$, ... Using the duality relations (A8), (A9), and (A12) we obtain

$$\frac{1}{(q-1)}\langle (q\delta\tau_{\tilde{i},1}-1)(q\delta\tau_{\tilde{j},1}-1)\rangle_{\tilde{K},h}$$

$$= \frac{1}{Z_{K,K_p}\{r\equiv 0\}}(Z_{K,K_p}\{r_p=-r_q;\, r_s=0;\, s\neq p, q\} + (q-2)Z_{K,K_p}\{r_p\neq -r_q,\, r_s=0;\, s\neq p, q\}). \tag{A15}$$

The value of Z_K does not depend on the absolute values of r_p, r_q, but only on $\delta(r_p+r_q)$ and $\delta(r_p)$, $\delta(r_q)$, where the δ is the Kroenecker delta.

Explicitly recalculating the graphical expansion for

$$\left\langle \sum_{\Gamma} (q\delta\tau_{\tilde{i},\Gamma}-1)(q\delta\tau_{\tilde{j},\Gamma}-1)\right\rangle = q\langle q\delta_{\tau_i,\tau_j}-1\rangle$$

we obtain the result given in (9).

(3) The three-point correlation function is found to give,

$$\frac{1}{(q-1)}\left\langle \prod_{l=\tilde{i},\tilde{j},\tilde{k}} \sum_{\mu=1}^{q} (S_l^{*}\Gamma_l)^{\mu}\right\rangle$$

$$= \frac{1}{Z_{K,K_p}\{r\equiv 0\}}$$

$$\times[(q-2)Z_{K,K_p}\{r_p+r_q+r_s=0;\ r_p+r_q\neq 0,\ r_q+r_s\neq 0,\ r_p+r_s\neq 0\}$$
$$+(q-1)(Z_{K,K_p}\{r_p+r_q+r_s\neq 0;\ r_p+r_q=0\}+\text{permutations})$$
$$+(q^2-6q+6)Z_{K,K_p}\{r_p+r_q+r_s\neq 0;\ r_p+r_q\neq 0,\ r_s+r_q\neq 0,\ r_p+r_s\neq 0\}],$$
$$(r_p,\ r_q,\ r_s\neq 0,\ r_t=0,\ t\neq s).$$

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